Finite Math - Fall 2018

Lecture Notes - 9/4/2018

Homework

- Section 2.5 - 1, 3, 5, 31, 34, 67, 68

Section 2.5 - Exponential Functions

Definition 1 (Exponential Function)**.** *An* exponential function *is a function of the form*

$$f(x) = b^x, \; b > 0, \; b \neq 1.$$

b is called the base*.*

Why the restrictions on b?

- If $b = 1$, then $f(x) = 1^x = 1$ for all x values. Not a very interesting function!
- As an example of the case when $b < 0$, suppose $b = -1$. Then

$$f\left(\frac{1}{2}\right) = (-1)^{1/2} = \sqrt{-1} = i$$

  an imaginary number! This kind of thing will always happen if b is negative.

- If $b = 0$, then for negative x values, f is not defined. For example,

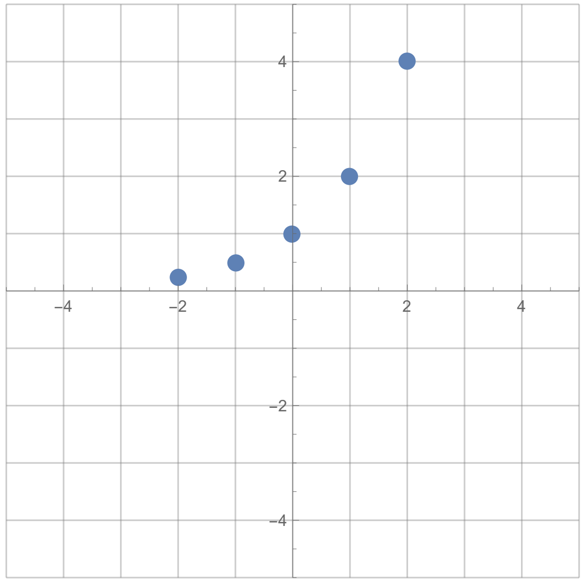
$$f(-1) = 0^{-1} = \frac{1}{0} = undefined.$$

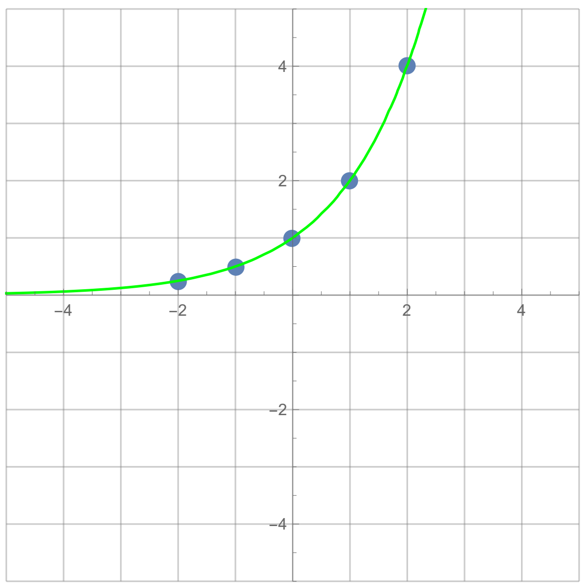
Let's get an idea of what these functions look like by graphing a few of them.

Example 1. *Sketch the graph of $f(x) = 2^x$.*

Solution. *Let's just plug in a few test points and connect the dots.*

x	-2	-1	0	1	2
$f(x)$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4



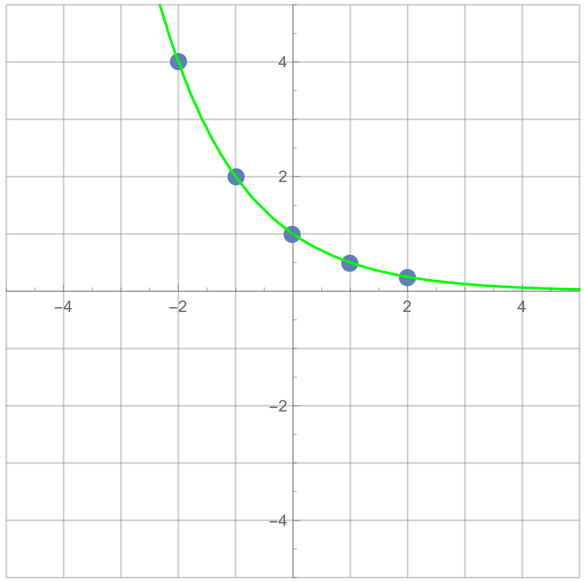


When $b > 1$, the graph of $f(x) = b^x$ has the same basic shape as 2^x, but may be steeper or more gradual. Let's see what happens when $b < 1$.

Example 2. *Sketch the graph of* $f(x) = \left(\frac{1}{2}\right)^x$.

Solution. *Let's just plug in a few test points and connect the dots.*

x	-2	-1	0	1	2
$f(x)$	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$



Notice that

$$\left(\frac{1}{2}\right)^x = (2^{-1})^x = 2^{-x}$$

so that when $b < 1$, we can set $b = \frac{1}{c}$ and have $c > 1$ and

$$f(x) = b^x = \left(\frac{1}{c}\right)^x = c^{-x}.$$

So, we can always keep the base larger than 1 by using a minus sign in the exponent if necessary.

Properties of Exponential Functions.

Property 1 (Graphical Properties of Exponential Functions)**.** *The graph of $f(x) = b^x$, $b > 0$, $b \neq 1$ satisfies the following properties:*

(1) *All graphs pass through the point $(0, 1)$.*

(2) *All graphs are continuous.*

(3) *The x-axis is a horizontal asymptote.*

(4) *b^x is increasing if $b > 1$.*

(5) *b^x is decreasing if $0 < b < 1$.*

Property 2 (General Properties of Exponents)**.** *Let $a, b > 0$, $a, b \neq 1$, and x, y be real numbers. The following properties are satisfied:*

(1) $a^x a^y = a^{x+y}$, $\dfrac{a^x}{a^y} = a^{x-y}$, $(a^x)^y = a^{xy}$, $(ab)^x = a^x b^x$, $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

(2) $a^x = a^y$ *if and only if* $x = y$

(3) $a^x = b^x$ *for all x if and only if* $a = b$

A Special Number: e. There is one number that occurs in applications a lot: the natural number e. One definition of e is the value which the quantity

$$\left(1+\frac{1}{x}\right)^x$$

approaches as x tends towards ∞.

This number often shows up in growth and decay models, such as population growth, radioactive decay, and continuously compounded interest. If c is the initial amount of the measured quantity, and r is the growth/decay rate of the quantity ($r > 0$ is for growth, $r < 0$ is for decay), then the amount after time t is given by

$$A = ce^{rt}.$$

Example 3. *In 2013, the estimated world population was* 7.1 *billion people with a relative growth rate of* 1.1%*.*

(a) *Write a function modeling the world population t years after 2013.*

(b) *What is the expected population in 2015? 2025? 2035?*

Solution.

(a) *We will write our function to output in billions. We will treat $t = 0$ as the year 2013, so that the initial population for this model is $c = 7.1$. The relative growth rate is 1.1%, which we must convert to a decimal before using $r = 0.011$. t will measure the years since 2013. Plugging these into the model, we get*

$$\textit{Population} = P = 7.1e^{0.011t} \textit{ billion.}$$

(b) *To find the estimated population in these years, we just need to plug the appropriate t-value into the model above.*

2015) $t = 2$

$$P = 7.1e^{0.011(2)} = 7.1e^{0.022} \approx 7.26 \textit{ billion}$$

2025) $t = 12$

$$P = 7.1e^{0.011(12)} = 7.1e^{0.132} \approx 8.1 \textit{ billion}$$

2035) $t = 22$

$$P = 7.1e^{0.011(22)} = 7.1e^{0.242} \approx 9.04 \textit{ billion}$$

Example 4. *The population of some countries has a relative growth rate of* 3% *per year. Suppose the population of such a country in 2012 is* 6.6 *million.*

(a) *Write a function modeling the population* t *years after 2012.*

(b) *What is the expected population in 2018? 2022?*

Solution.

(a) $P = 6.6e^{0.03t}$

(b) 7.9 *million;* 8.9 *million*